Using tiling theory to generate angle weaves with beads

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Abstract

Tilings of the plane, especially periodic tilings, can be used as the basis for flat bead weaving patterns called angle weaves. We describe specific ways to create intricate and beautiful angle weaves from periodic tilings, by placing beads on or near the vertices or edges of a tiling and weaving them together with thread. We also introduce the notion of star tilings and their associated angle weaves. We organize the angle weaves that we create into several classes, and explore some of the relationships among them. We then use the results to design graphic illustrations of many layered patterns. Finally, we prove that every normal tiling induces an angle weave, providing many opportunities for further exploration.

Keywords: periodic tiling of the plane, tessellation, angle weave, RAW, seed bead weaving, jewelry, star tiling, needlework, fiber art
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1. Introduction

Bead weavers create a wide variety of designs by joining beads (any object with a hole) with needle and thread, including flat weaves that resemble woven fabric (see Figure 1). Patches of beaded fabric made from beads as tiny as poppy seeds (thereby known as *seed beads*) are commonly used to make jewelry, especially bracelets and necklaces. For both aesthetic and practical reasons, beaded fabric is often designed with a pattern that can be repeated to cover an arbitrarily large region in a visually appealing way. This provides a natural connection with the mathematical theory of tilings of the plane. However, while tilings of the plane have been studied by artists and mathematicians for centuries, relatively few have been used to inspire beading designs. In this paper, we will explore beading patterns arising from periodic polygonal tilings of the plane (which we call *angle weaves*), and show that this is merely the tip of the iceberg of all the possible beading designs that can be inspired by mathematical tilings.

In Section 2, we will review ideas and notation from the mathematical theory of tilings. We will also introduce the new notion of a *star tiling*. In Sections 3, 4 and 5 we will proceed to explore some of the intricate and beautiful beading patterns that can be derived from polygonal tilings, providing examples and illustrations as we go. We classify several different methods for creating these patterns, and explore some of the relationships between them. In Section 6, we will prove that an infinite class of tilings, the *normal tilings*, can be used as the basis for angle weaves, providing the opportunity for much more exploration and investigation in the future.

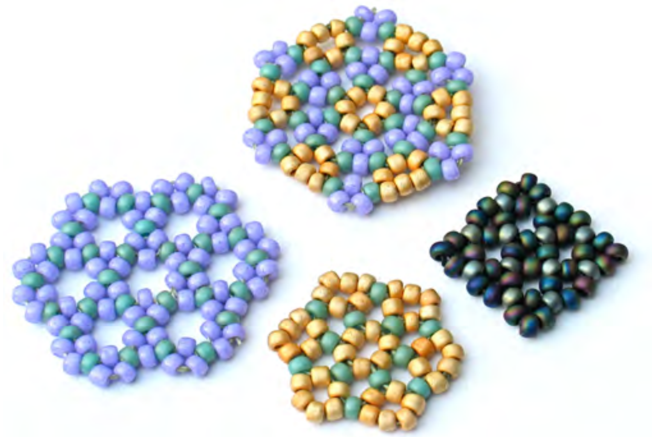


Figure 1: Flat weaves based on repeating tilings of the plane (clockwise from top): Snow Star with only vertex beads, super RAW, hexagon angle weave, Archimedes' Star with only vertex beads (See also the section on across-edge weaves)

2. Tilings of the plane

In this section we will briefly review definitions and notation from the mathematical theory of tiling; for more information we refer the reader to [11] and [12]. We will also introduce the idea of a *star tiling*, which will be the source of many of our beading patterns.

2.1. *Normal polygonal tilings*

A *tiling* (or *tessellation*) of the plane is any way of covering the plane with finite shapes (tiles) so that there are no gaps and no two tiles overlap (so tiles only touch at their corners and along their sides). A tiling is *periodic* if it is possible to overlay the tiling with a grid of parallelograms so that the pattern inside each parallelogram is the same. The patch of the tiling inside one parallelogram is called a *translational unit,* and it can be used to reconstruct the entire pattern by repeated translations by the vectors that define the grid. We are primarily interested in tilings where the individual tiles are polygons, since these are the kinds of tilings that we can weave with beads. The sides of the polygonal tiles are called *edges* of the tiling, and the corners of the polygonal tiles are called *vertices* of the tiling. Figure 2 shows three examples of periodic polygonal tilings with circles (representing beads) placed on the midpoint of each edge.

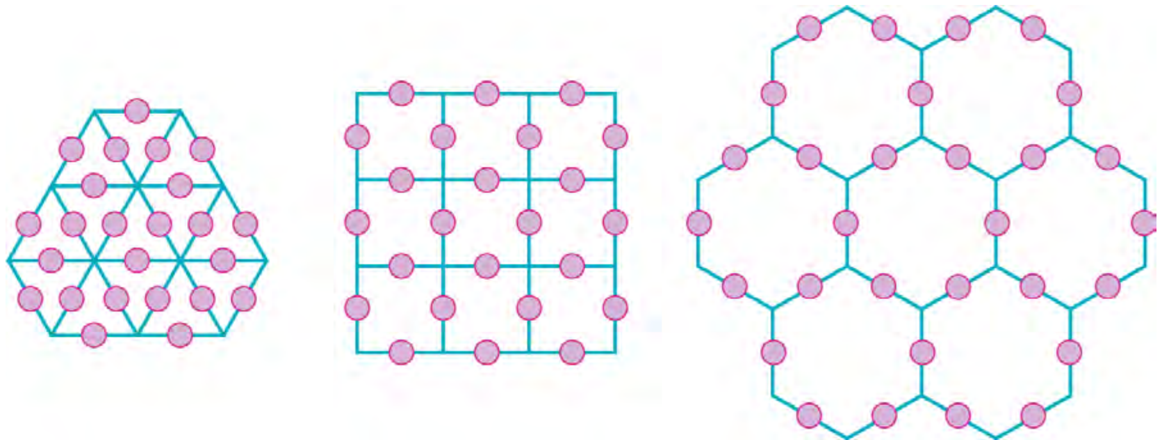


Figure 2: The regular tilings with one bead on each edge: Triangle weave (3^6), right angle weave (4^4), and hexagon angle weave (6^3)

We need to add another technical restriction to our tilings. Consider an infinite nested set of similar polygons, with extra edges added to connect the corresponding vertices of the polygons, as in Figure 3. Then a patch containing the singular point contains an infinite number of edges and vertices. Since our angle weaves will be constructed by sewing beads onto edges and/or vertices, this is a problem!

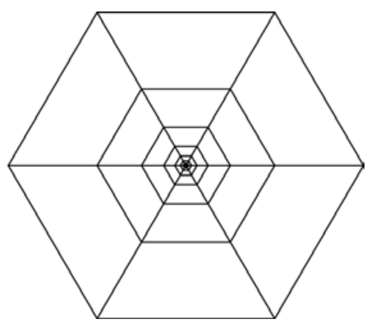


Figure 3: A tiling that is not locally finite

Another problem arises when the tiles have holes, as in Figure 4 (left), or if two tiles touch in disjoint arcs, as in Figure 4 (right). In this first case, the beaded fabric would simply fall apart; in the second, part of the pattern would spin independently from the rest when woven.

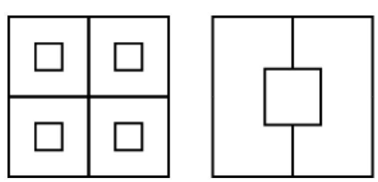


Figure 4: A tiling with tiles that are not simply connected, tiles that touch on two disjoint edges

To avoid these problems, we restrict ourselves to *normal tilings* (see [11, section 3.2]). A tiling is *normal* if (1) every tile is a topological disk (i.e., it has no holes as in Figure 4 (left), and the loop around its outside doesn't intersect itself), (2) the intersection of every two tiles is a connected set (so tiles cannot touch in two or more disjoint edges as in Figure 4 (right)) and (3) the tiles are uniformly bounded (there are minimum and maximum sizes for the tiles, unlike Figure 3). In this paper, the term *tiling* will refer only to normal polygonal tilings, usually periodic ones.

2.2. *Regular, Archimedean and dual tilings*

The *valence* of a vertex of a tiling is the number of edges that meet at that vertex or the number of tiles that meet at the vertex (these are equivalent for normal tilings). In a tiling where all tiles are regular polygons, the *type* of a vertex is the number of sides of the tiles incident to the vertex, listed in cyclic order around the vertex. So if a vertex is incident to tiles with n_1, n_2, ..., n_k sides, its type is given by the symbol $n_1.n_2.n_3....n_k$. Since there are many ways to do this (depending on which tile we start with, and which direction we go around the vertex), the standard practice is to take the symbol that is lexicographically first among all the possibilities. We use exponential notation to indicate when a number is repeated several times in succession; for example, 3.12.12 can also be written as 3.12^2. If all the vertices of a tiling are the same type, we denote the tiling by $(n_1.n_2.n_3....n_k)$; such tilings are called *Archimedean*, and there are exactly 11 of them (see [11, section 2.1]). These include the three *regular* tilings (3^6), (4^4) and (6^3), which are all the tilings of the form (p^r) (see Figure 2). Tilings with vertices of k types, such that any vertex can be taken to any other vertex of the same type by a symmetry of the tiling, are called *k-uniform*, and denoted by the symbol $(a_1.b_1.c_1...; a_2.b_2.c_2...; ...; a_k.b_k.c_k...)$.

Given a tiling by regular polygons T, we define its *dual tiling* T^* by placing a vertex in the center of each tile of T, and connecting them by edges perpendicular to the edges of T (see, for example, [12, Section 4.2]). For example, Figure 5 shows that the dual tiling of (4^4) is (4^4) (so this tiling is *self-dual*), and the dual tiling of (6^3) is (3^6) (and vice versa).

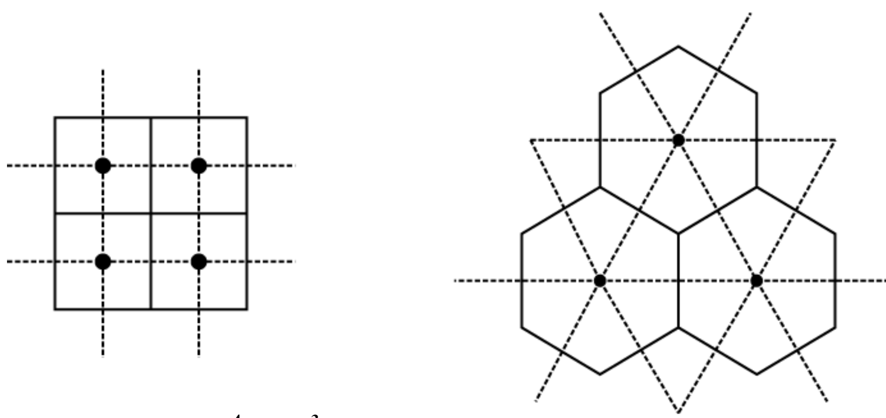


Figure 5: The tilings 4^4 and 6^3, with their dual tilings (shown with the dashed lines)

The duals of the Archimedean tilings are the 11 *Laves* tilings (see [12, section 4.2]). Since the tiles of the Archimedean tilings are regular polygons, the vertices of the Laves tilings are also regular, meaning that if v edges meet at a vertex, the angle between each consecutive pair of edges is $360°/v$. We denote each Laves tiling by the symbol for its dual Archimedean tiling.

2.3. *Star Tilings*

A *star tiling* consists of multiple copies of stars, such as those in Figures 6 and 7, joined to create a periodic tiling. Any polygonal tiling can be used to construct a star tiling.

Figure 6: Stars with 3, 4, 6 and 8 points

A *star* with $n \geq 3$ points is a set of n triangles, meeting corner to corner so that their bases form a polygon with n sides (with the triangles all pointing outwards). The *n points* of a star are the n apexes of the n triangles. A star is *regular* if all of its triangles are equilateral and congruent and the interior polygon is regular (see Figure 7). A star is *semiregular* if the interior polygon is regular and the triangles are all isosceles (but not necessarily equilateral, or congruent) with bases on the interior polygon. The star in Figure 6 with 8 points is semiregular.

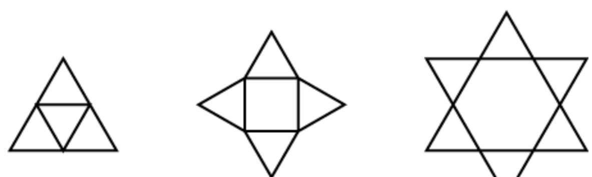


Figure 7: Regular stars with 3, 4 and 6 points

To create a star tiling from an initial polygonal tiling, we place a star at each vertex so that (a) the number of points on each star is equal to the valence of the associated vertex of the initial tiling, (b) the centers of the stars are aligned with the vertices of the initial tiling, and (c) the points of the stars land on the midpoints of each edge of the initial tiling. Examples of this transformation on the three regular tilings are shown in Figure 8.

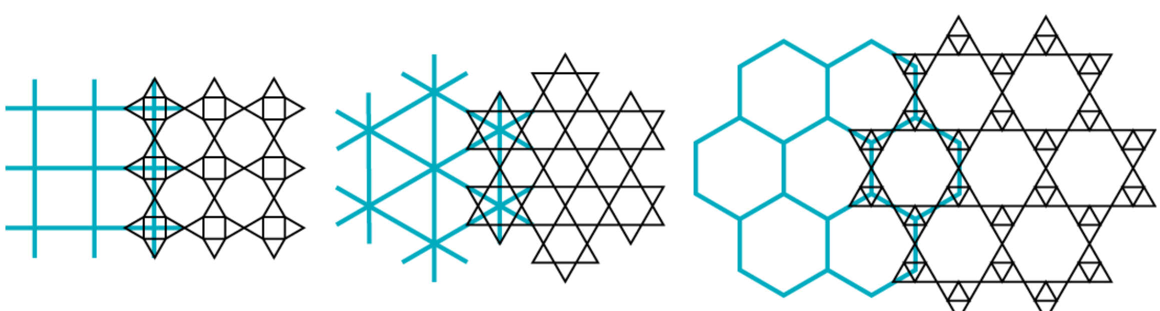


Figure 8: Star tilings of (4^4), (3^6), and (6^3): Kepler's Star, David's Star, and Archimedes' Star

A given tiling may have many different star tilings, depending on how we choose the interior polygon in each star. In this paper we will choose the stars to be as regular as possible (though this is not necessary in general). A star tiling is *regular* if all the stars are regular and congruent. There are three regular star tilings, based on the three regular tilings of the plane (4^4), (3^6) and (6^3). The star tiling based on (3^6) is more commonly known as the Archimedean tiling (3.6.3.6); since the 6-pointed stars resemble the Star of David, we call this tiling (and its associated weaves) *David's Star*. The star tiling based on (6^3) also closely resembles the tiling (3.6.3.6), except with an additional triangle inscribed inside each triangle of (3.6.3.6); we call this *Archimedes' Star*. Our drawing of the regular star tiling based on (4^4) is not a tiling by regular polygons (the octagons are not regular), but it resembles the 2-uniform tiling (3.4.3.12; 3.12^2). That tiling was published by the astronomer and mathematician Johannes Kepler in 1619 [11], so we call a star tiling of (4^4), *Kepler's Star*, in honor of his pioneering work in tiling theory.

A star tiling is *semiregular* if all the stars are semiregular. Any tiling in which the vertices are regular induces a semiregular star tiling, since then the edges incident to each vertex are perpendicular bisectors of the edges of a regular polygon centered at that vertex. In particular, we can consider the 11 Laves tilings mentioned in the last section. In addition to the three regular star tilings, two semiregular star tiling based on a Laves tiling are Snow Star and Night Sky. Snow Star is the star tiling of the tiling dual to (3.6.3.6), that is, the tiling by 60° diamonds (known to quilters by the name of Baby Block or Tumbling Blocks) (see Figure 9, left). The Snow Star is composed of regular stars with three points and six points. Night Sky is the star tiling of the tiling dual to (4.8^2), the tiling by isosceles right triangles in Figure 9, right.

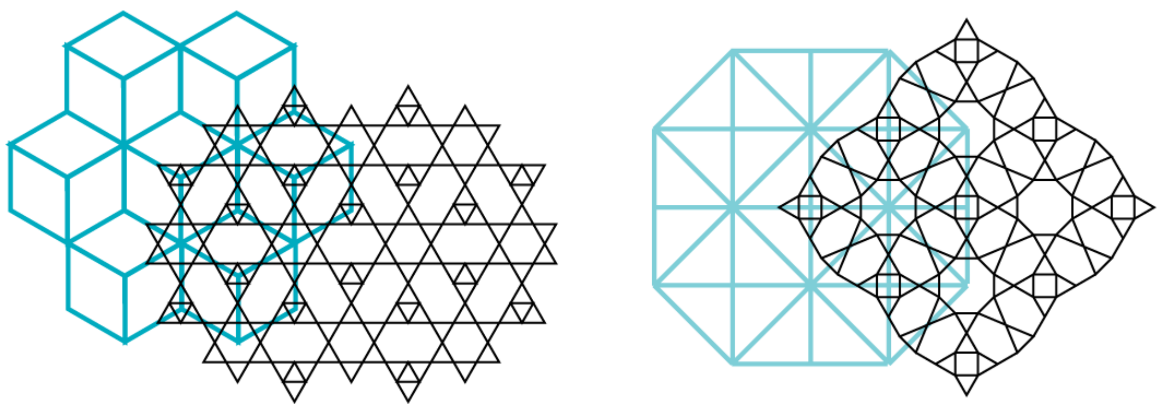


Figure 9: Examples of Laves tilings and their star tilings: Snow Star and Night Sky

3. Weaving beads on an edge: edge-only and edge-and-cover angle weaves

How do we turn a tiling into a beading pattern? In general, an *angle weave* of a tiling is a weave used to join together beads arranged on (or near, as in Figure 12) the edges and/or vertices of the tiling so that the beads on each tile in the tiling are connected in order (e.g., in a loop). The most obvious way to do this is to place a bead on every edge of the tiling so that the hole of the bead is aligned with the corresponding edge of the tiling. This is called an *edge-only angle weave.* Figure 2 shows edge-only bead weave diagrams of the three regular tilings, and Figure 10 shows these same patches woven with real beads and thread.

Figure 10: Examples of triangle weave (3^6), right angle weave (4^4), and hexagon angle weave (6^3) with fire-polished 4mm beads

A weave is constructed by passing thread through each bead so that the beads surrounding each tile of the pattern are connected with thread. The beads around each tile are generally sewn in a loop, but the tilings in Figure 2 suggest no particular thread path, and in fact, many different thread paths are possible. In practice, the beads may be of varying sizes, or there may be multiple beads on a single edge. Also, different tilings can admit the same edge-only weave; for example, many different tilings by quadrilaterals admit the right angle weave shown in Figure 10, center. This paper describes five classes of angle weaves, depending on where the beads are placed: edge-only, edge-and-cover, vertex-only, vertex-and-edge, and across-edge angle weaves. We begin with the three simplest examples of angle weaves, the regular edge-only angle weaves in Figures 2 and 10.

The edge-only weave of the tiling (4^4) is commonly called *right angle weave*, or *RAW* for short, and it is the most popular of all of the angle weaves. A Google™ search for "right angle bead weave" yields hundreds of thousands of hits. Moreover, dozens, if not hundreds of books and articles have been written on RAW, such as those by Chris Prussing [17] and Marcia DeCoster [1]. RAW is so popular that some authors and publishers refer to all angle weaves as variations of RAW, regardless of whether or not the underlying tiling has any right angles in it. We prefer the more general term "angle weave" of which we view RAW as a special case.

Less common than RAW, but also used by bead weavers, is the regular *triangle weave*, which corresponds to the tiling (3^6). Instructions for the triangle weave are provided by [15] using two needles, and [16] using a single needle. The *hexagon angle weave* based on the regular tiling (6^3) is less popular that the other two regular weaves, but is still often used, and has been studied by artists such as Lenz [13]. Hexagon angle weave is quick and easy to execute, and it is easy to weave in several different directions [2, 3]. Figure 11 shows two examples of bracelets woven using the hexagon angle weave. These two bracelets show how the same angle weave can look different when different numbers and shapes of beads are placed on different edges of the tiling. Traditional Zulu beaders use a netting weave that produces hexagons [10], but beaders tend to classify netting weaves as distinct from angle weaves because of how they are constructed, even though the arrangement of beads may be the same; nettings are sewn with a zig-zag stitch while angle weaves are sewn with loops. Of the many designers who use angle weaves, Gerlinde Lenz [13] and Laura Shea [18] are particularly well known.

Figure 11: Hexagon weave with size 10° and 11° seed beads, hexagon weave with size 11° seed beads and bugle beads

In more elaborate designs, there are often additional beads woven around the vertices. In the context of beaded polyhedra (such as the example in Figure 15), Lenz [14] calls the edge beads *structural beads*, since they give the weave its underlying pattern; Prussing [17] calls them *crossing* or *working beads*. Other beads are woven between the edge beads to cover the thread, provide decoration, and stabilize the corners [14]. These *cover beads* are also called *stabilizing beads* [14], *in-between beads*, or *thread covers* [17]. This gives rise to the notion of an *edge-and-cover* angle weave of a tiling as in Figure 12. The tiling is in gray, the thread is in black, the edge beads are large, and the cover beads are small.

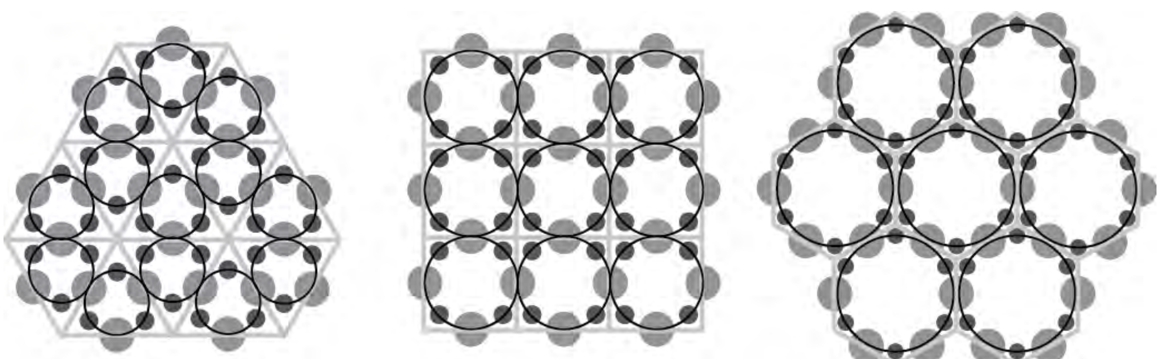


Figure 12: Edge-and-cover angle weaves for (3^6), (4^4), (6^3)

In an edge-and-cover angle weave of a tiling, each edge of the tiling has one bead and every vertex with valence n has n beads, arranged in order around that vertex. Thus, in this type of angle weave, each tile has one bead on each side and one bead near each corner, as shown in Figure 12. So a tile with m sides corresponds to a loop of $2m$ beads. In practice, edge-and-cover angle weaves can be stiff or flexible depending upon the choice of pattern and bead sizes. If a flexible fabric is woven, the cover beads provide nice anchors to weave another layer of beads on top of the first (as in the beaded polyhedron in Figure 15).

4. Weaving beads at a vertex: vertex-only and vertex-and-edge angle weaves

Thus far we have discussed edge-only and edge-and-cover angle weaves, created by placing beads on edges of a tiling, and possibly near a vertex (i.e., as cover beads). Now we will consider the problem of placing a bead directly on a vertex. If the valence of the vertex is three or more, then there are several ways to orient the bead hole and connect the bead to its neighbors. However, we will only explore placing beads on the vertices of the star tilings introduced in Section 2.3; in this case, every vertex has valence four. We choose to always orient a vertex bead so that it is adjacent to two edge beads at each end of its hole. While the orientation of the hole of any edge bead is unique, there are exactly two ways to orient a vertex bead with valence four (see Figure 13), and either orientation is weavable.

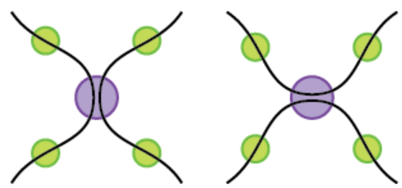


Figure 13: Two ways to orient a vertex bead

For the beading patterns created by weaving star tilings, we stitch a loop connecting the beads in each polygon, except for the triangles that form the points of the stars. For the triangles, notice that every vertex of a star tiling is a point where two triangles meet (not counting the interior polygon, if it happens to be a triangle). We choose to orient the vertex bead holes to point to the centers of these two triangles. Figure 14 shows how we arrange one bead at each vertex of a triangle. We choose this particular arrangement because it hides the bead holes and shows little or no thread. This arrangement can be generalized for other polygons, but in practice, the triangle usually gives the tightest weave which is why an extra loop of thread around the triangles is omitted.

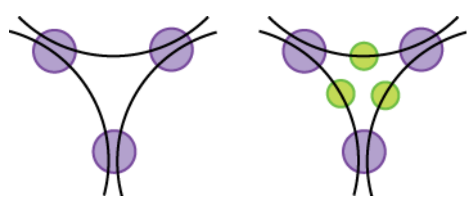


Figure 14: Orientation of beads on a triangle: on vertices only and on both vertices and edges

5. Star weaves

Star tilings grew out of previous work on beaded beads, specifically the Octahedral Cluster (see Figure 15, left) [8]. To make an Octahedral Cluster, we start with an edge- and cover weave of a regular octahedron (Figure 15, right). We then weave an outer layer of beaded stars that are connected to the octahedron by the cover beads.

Figure 15: Octahedral Cluster Beaded Bead, Edge-and-cover beaded regular octahedron

Inspired by these beaded stars, we created star tilings to design flat weaves for bracelets and flat pendants. A *star weave* is a beading pattern arising from a star tiling.

A star weave is generated in a two-step process. We start with an arbitrary tiling, which is transformed into a star tiling as described in Section 2.3. The star tiling is then transformed into an angle weave in one of three distinct ways, by placing one or more beads on each edge (as in Section 3) and/or each vertex (as in Section 4) of the star tiling. Just as an edge-only angle weave of a tiling only has beads at the edges of the tiling, a vertex-only angle weave only has beads at the vertices, and a vertex-and-edge angle weave has beads at both vertices and edges. Figure 16 shows the Kepler's Star tiling with beads on every edge (left), with beads on every vertex and edge (center), and beads on every vertex (right).

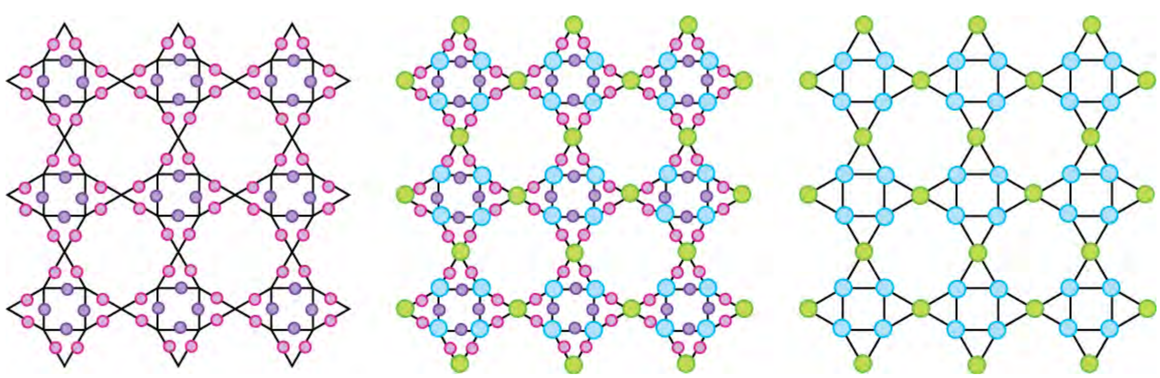


Figure 16: Kepler's Star with beads on edges only, vertices and edges, and vertices only (super RAW)

When woven in beads, the edge-only version makes for a saggy weave that shows thread and the holes of the beads. This is often true for edge-only weaves. So, for aesthetic reasons, we focus on the other two possibilities. For Kepler's Star Weave, using the tightly packed and more elegant version with beads on both vertices and edges, as shown in the center of Figure 16, resulted in the beaded bracelet in Figure 17 [4]. Of course, the beads in Figure 16 are idealized and can be replaced with larger, smaller, or more beads. For example, the bracelet in Figure 17 contains two sizes of seed beads, with two small beads used along two edges of each triangle. The design is further enhanced by carefully choosing bead colors to create an illusion of linked rings. Happily, the four-pointed stars in the Kepler's Star bracelet resemble the stars in the Octahedral Cluster Beaded Bead (Figure 15, left). So the vertex-and-edge weave of this star tiling achieves our goal of making a flat weave of beaded stars.

Figure 17: Kepler's Star Bracelet with size 11° and 15° seed beads

Now consider the vertex-only weave shown on the right of Figure 16; we call this *Super RAW*, and [5] provides a tutorial. Compare this to the edge-and-cover weave of (4^4) (see Figure 12, center). The beads in both weaves are in the same relative positions, but the thread paths are different. In particular, Super RAW has extra passes of the thread that connect the would-be-cover beads together in a loop at each corner of (4^4), the original tiling used to generate both cases. Similar relationships are true for other tilings as well, meaning we can generate the same beading pattern in more than one way. Sometimes we get exactly the same pattern, as we describe later in Theorems 2 and 3. Other times, like this one, the two methods result in the same bead arrangement but one method includes extra thread connecting some of the beads. This is the essence of our first theorem. Moreover, if there are more than three cover beads at each vertex, as in the example for (4^4), the extra passes of the thread make a noticeable difference in the fit of the resulting beaded fabric. In the case of just 3 cover beads at each vertex (e.g., as for (6^3)), the difference is not very noticeable to the eye or hand, but the extra thread takes longer to stitch and creates a stronger fabric.

<u>Theorem 1:</u> Let *T* be a tiling. The vertex-only weave of the star tiling of *T* has the same bead pattern as the edge-and-cover weave of *T*, plus extra thread that connects the cover beads at the vertices of *T*.

<u>Proof:</u> As we can see in Figure 18, the beads placed on an edge of *T* are also at the points of the corresponding star of *T*, in the same orientation. The beads placed near a vertex of *T* are in the same location and orientation as those placed around the center of the star in the star tiling of *T*, but in the star tiling, they are connected together in a loop. ■

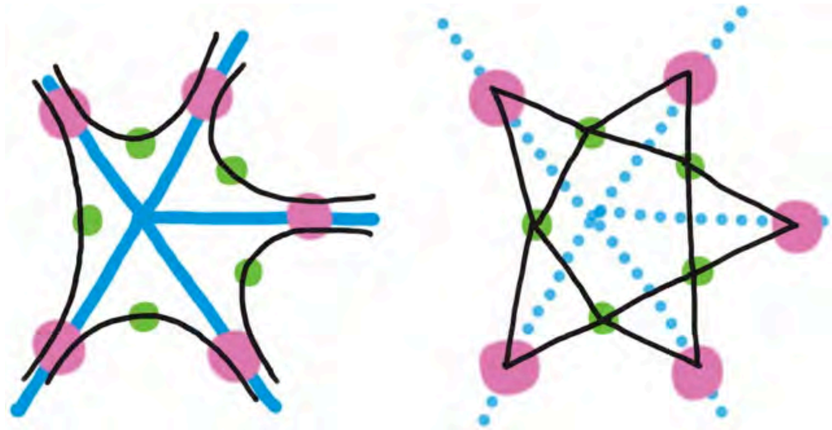


Figure 18: Edge and cover weave of T, Vertex only weave of star tiling of T

Figure 8 (center) shows how we used (3^6) to create the David's Star tiling (3.6.3.6), and Figure 19 shows how we place beads on this tiling using only vertex beads (left) and both vertex and edge beads (right). The pattern shown at the bottom of Figure 1 is the same as the pattern on the left in Figure 19 (with different boundary). The weave in Figure 1 is hexagon angle weave (the edge-only angle weave of (6^3)); we now see it is also the vertex-only weave of the star tiling of (3^6). This is no coincidence, and reflects the fact that (3^6) and (6^3) are dual tilings. Although the weave in Figures 1 (bottom) and 19 (left) uses two colors, all of the beads can be made the same type so that every loop has 6 identical beads as in Figures 2 and 10 (right). Since every vertex in (3.6.3.6) is the same type, so is every bead in the weave; that is, every bead has the same thread path and placement relative to the surrounding beads. Although it might seem cumbersome to think of hexagon angle weave (Figure 2, right) as the vertex-only weave of the star tiling of (3^6) (Figure 19, left), the star pattern is useful because it identifies two different edge types, which, when beaded along with the vertices, create the more intricate David's Star pattern (Figure 19, right).

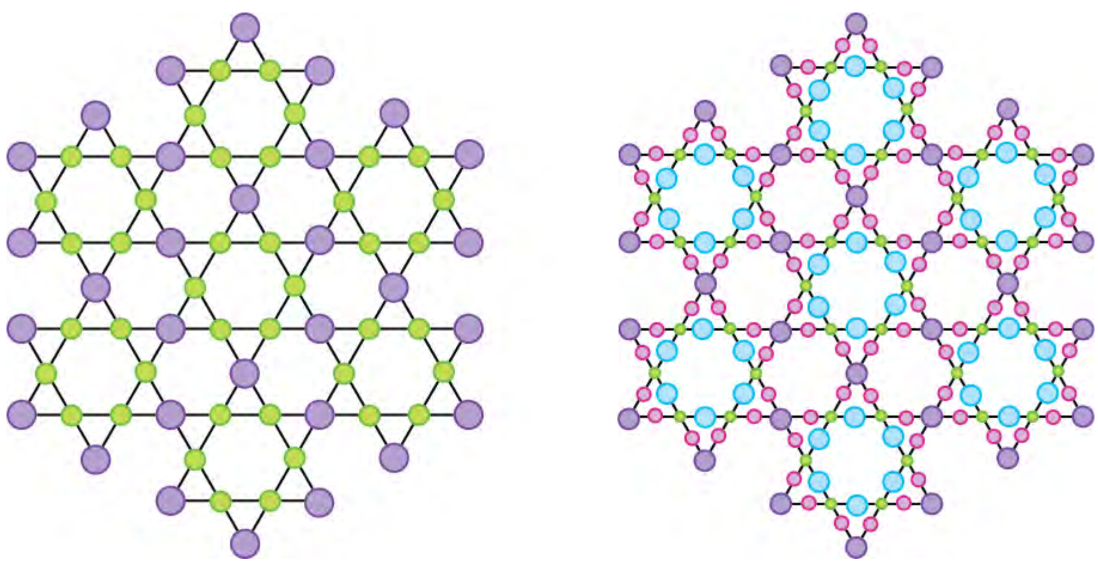


Figure 19: Hexagon angle weave, David's Star

David's Star weave is almost as easy to weave as hexagon angle weave. The three bracelets in Figure 20 show different examples of David's Star with real beads. The top left version is simplified to use just two types of beads. The top right bracelet takes advantage of the fact that select rows of beads in the weave can be enlarged without altering the rest of the weave. Many other weaves, such as Kepler's Star, have this same property. The bottom bracelet shows how bugle beads on some of the edges of the David's Star tiling (those colored pink in Figure 19, right) emphasize a star motif, giving star weaves their names.

Figure 20: David's Star Bracelets in size 11° and 15° seed beads; size 11° and 15° seed beads and 4mm crystals; size 8° and 11° seed beads and bugle beads

Now consider the star weaves of (6^3). Figure 8 (right) shows how we used (6^3) to create the Archimedes' Star tiling. Figure 21 shows how this tiling looks with only vertex beads (left) and with both vertex and edge beads (right). As predicted by Theorem 1, Archimedes' Star with only vertex beads (Figure 21, left) gives the same arrangement of beads as the edge-and-cover weave of (6^3) (Figure 12, right), but with additional loops of thread. For the bead weaver, the edge-and-cover weave of (6^3) is the more elegant of the two versions because it requires fewer passes of the needle to produce the same beaded fabric. However, the extra thread of Archimedes' Star allows for additional possibilities for the boundaries of woven patches, as shown in Figure 1, left. One can identify the thread path of this patch as Archimedes' Star by looking at the boundary. As we found with our previous examples of David's Star and Kepler's Star, the extra thread also leads to an entirely new weave when we use both vertex and edge beads. The result is Archimedes' Star weave (Figure 21, right), another tightly fitting weave with particularly large holes. Figure 22 shows an example of a bracelet woven using this pattern. The first author provides the step-by-step instructions for weaving this bracelet in [7].

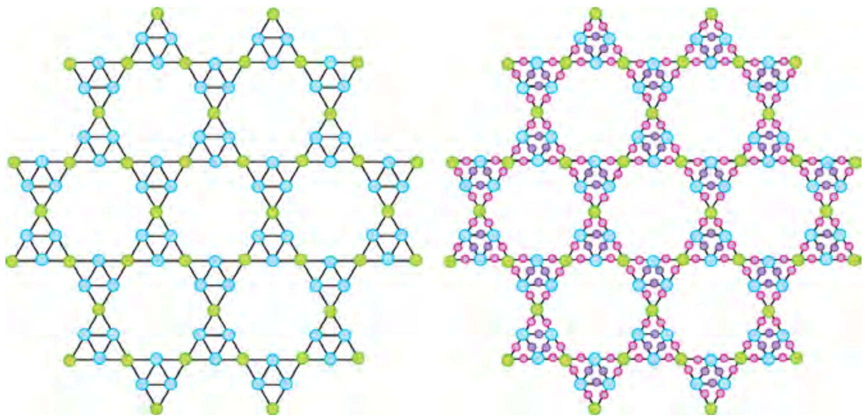


Figure 21: Archimedes' Star with only vertex beads, and with vertex and edge beads

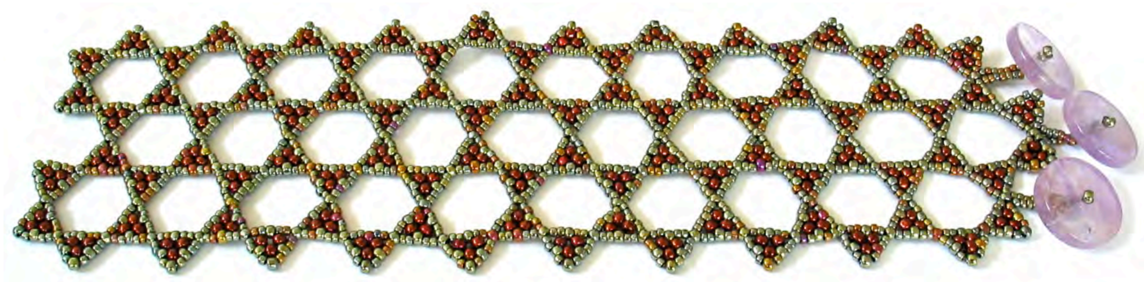


Figure 22: Archimedes' Star Bracelet with size 11° and 15° seed beads

We finish this section with a few examples of the star weaves generated from Laves tilings. The Night Sky tiling (Figure 9, right) induces two beautiful semiregular star weaves. Figure 23 shows the patterns obtained by putting beads at only the vertices (called the Picnic Weave because of its resemblance to a plaid picnic blanket), and at both the edges and vertices (called the Night Sky Weave). Examples of these weaves are shown in Figure 24. Note that the pendant in Figure 24 does not show the same patch of the tiling as shown on the left in Figure 23. The illustration for Night Sky on the right in Figure 23 is a subset of the patch used in the completed bracelet on the right in Figure 24. The first author provides the step-by-step instructions for weaving Picnic Weave and Night Sky Weave in [6].

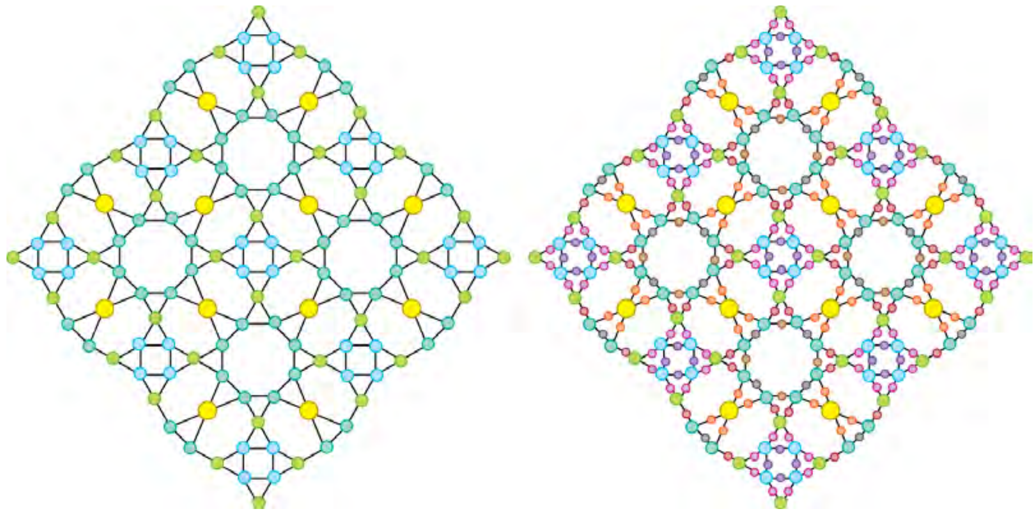


Figure 23: Picnic Weave (only vertex beads) and Night Sky (vertex and edge beads)

Figure 24: Picnic Pendant with size 8°, 11° and 15° seed beads; Night Sky Bracelet with size 8°, 11° and 15° seed beads

Figure 9 (left) shows how we used a Laves tiling to generate the semiregular Snow Star tiling. The Snow Star is composed of regular stars with three points and six points. Figure 25 shows how the Snow Star tiling can look as a star weave with both edge and vertex beads. Figure 1 (top) shows the Snow Star weave with vertex-only beads.

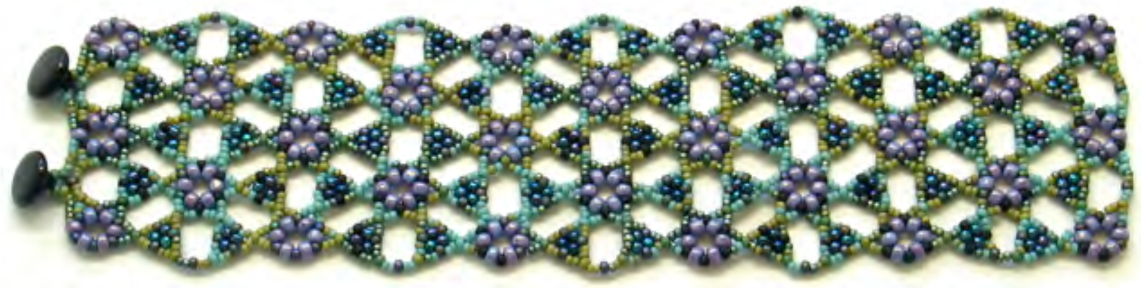


Figure 25: Snow Star Bracelet with size 8°, 11° and 15° seed beads

Figure 26 shows two examples of star weaves with only vertex beads using the Laves tiling (3.12^2) to generate the star tiling. This figure shows how changing the sizes and counts of the beads on each vertex can affect the texture of the resulting beaded fabric.

Figure 26: Star Weave of Laves tiling (3.12^2) with only vertex beads: size 8° and 11° seed beads; size 8°, 11° and 15° seed beads

6. Across-edge weaves

In this section we will introduce a fifth way to use a tiling to generate a bead weave. At first glance, this looks quite different from what we've seen so far, but we will discover that it is closely related to the star weave.

Given a tiling *T*, we create the across-edge weave pattern of *T* in three steps, illustrated in Figure 27 for the example of a hexagonal tile. We first place one bead on each edge of *T*, with the hole perpendicular to the edge (these are the rectangles in Figure 27). Then, for each tile in *T*, we place a bead between each pair of beads on adjacent edges, with the hole oriented towards the beads on the edges (these are the ellipses in Figure 27). Finally, the thread path for the weave (the dotted line in Figure 27) connects each edge bead to the two beads inside the tile that are adjacent to it, and also connects the two adjacent interior beads.

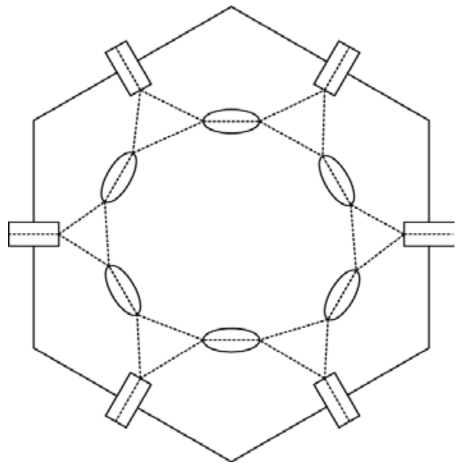


Figure 27: An across-edge weave for a hexagonal tile

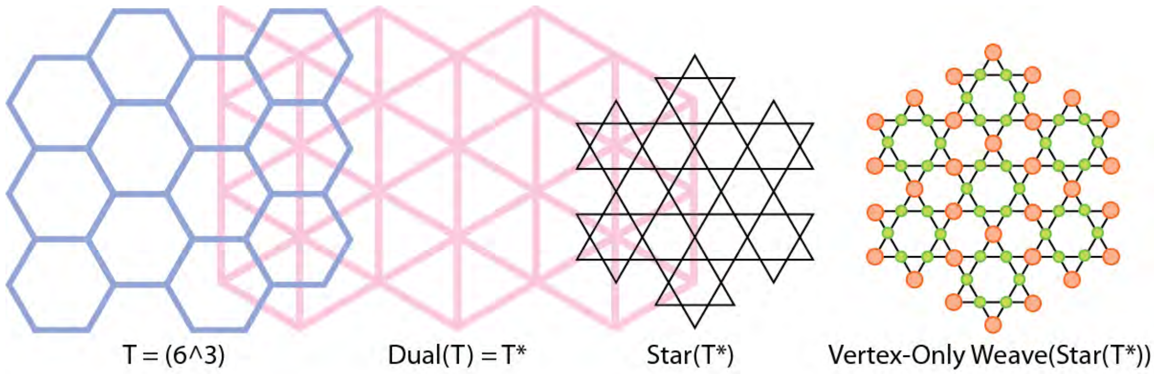
Figure 1 shows four examples of across-edge weaves in beaded fabric. Moving from the top clockwise, we have the across edge weaves of (3.6.3.6), (4^4), (6^3) and (3^6).

The next theorem describes the relationship between across-edge weaves and star weaves.

Theorem 2: Let *T* be a tiling, and *T** its dual tiling. Then the across-edge weave of *T* has the same pattern as the vertex-only star weave of *T**.

Proof: As we can see in Figure 27, the beads placed on the interior of a tile of *T* are on the vertices of a polygon with the same number of edges as the original tile; this number is the valence of the corresponding vertex of *T**. The beads on the edges of *T* are then at the points of the triangles whose bases form the edges of the polygon. Together, these are all the vertices of a star centered at a vertex of *T**. ■

Figure 28 illustrates Theorem 2 for $T = (6^3)$. A beautiful consequence of Theorem 2 is that we can draw the sequence of overlapped patterns all in one picture. Figure 29 shows how such a drawing might look.



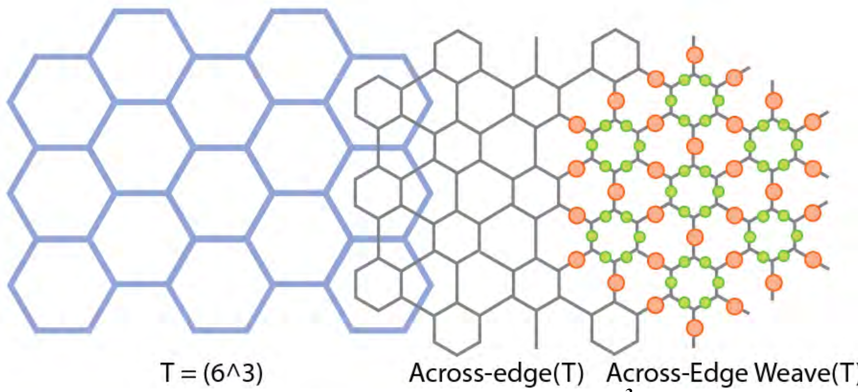


Figure 28: Theorem 2 for $T = (6^3)$

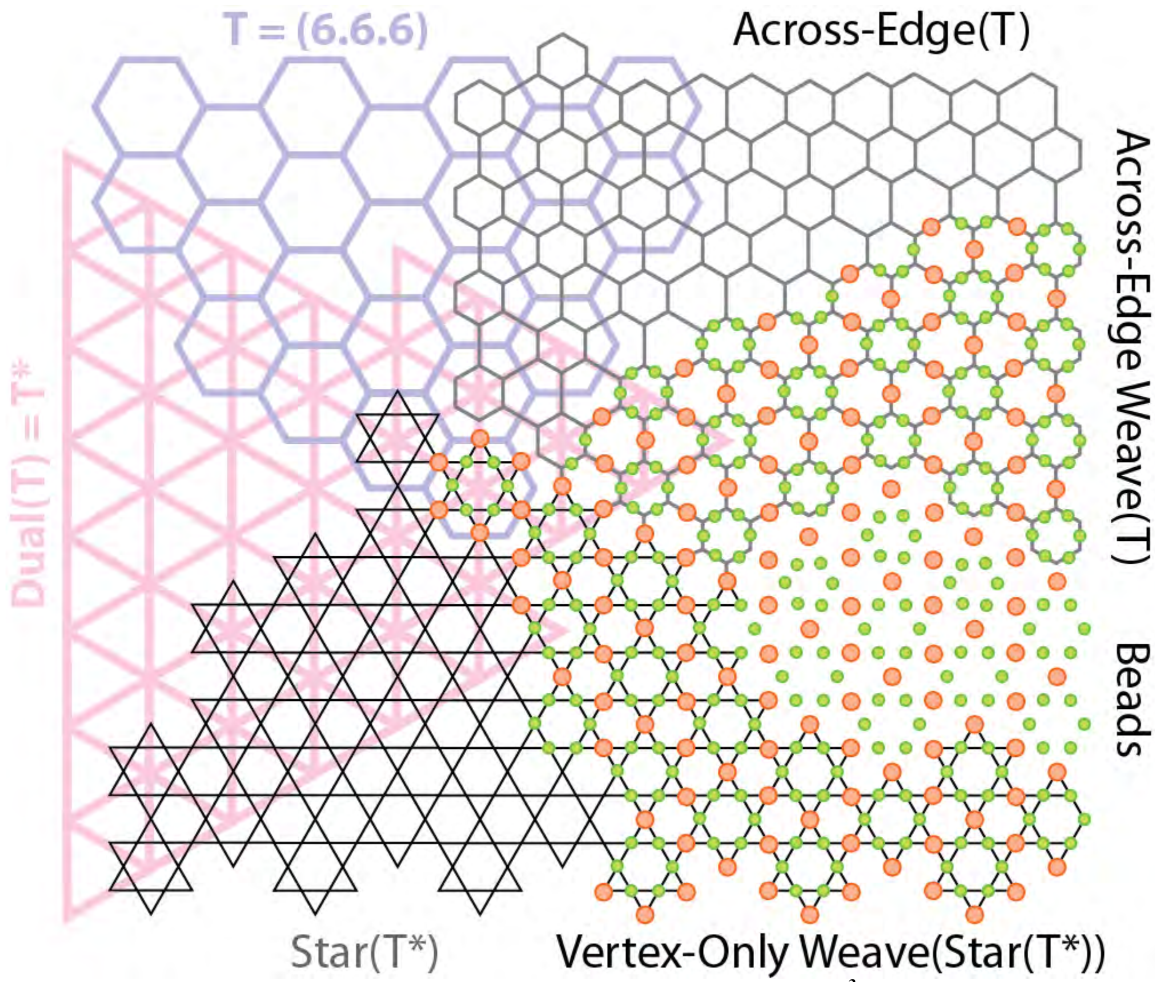


Figure 29: Design based on Theorem 2 for T = (6^3)

Figure 30 illustrates Theorem 2 for $T = (3^6)$. Figures 29 and 30 can be viewed as duals of each other.

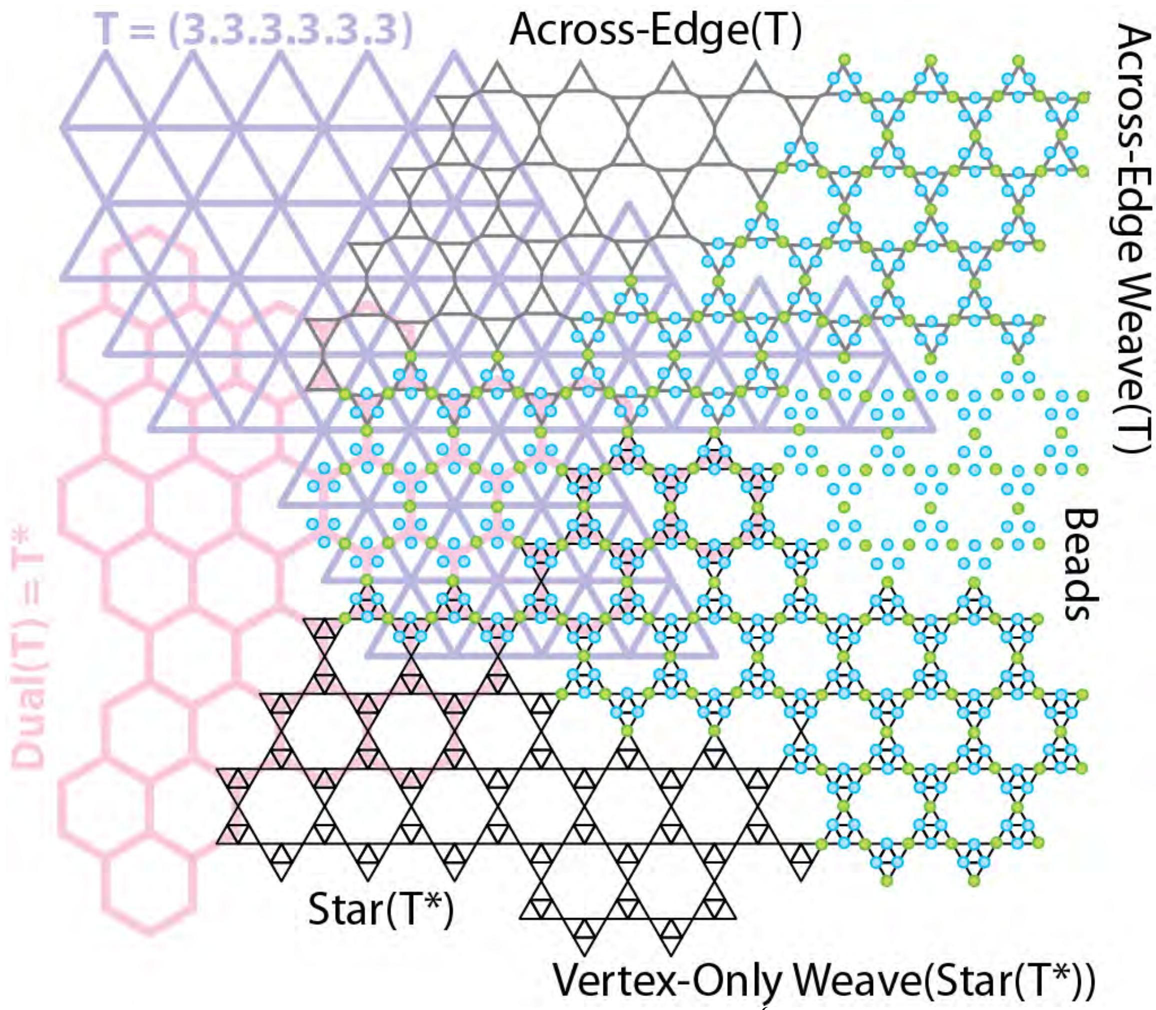


Figure 30: Theorem 2 with $T = (3^6)$

The correspondence between across-edge weaves and star weaves can be generalized further. For any single tiling, you can imagine an across-edge weave where there are two beads (on two separate thread paths) across every edge, and an extra bead between them inside the polygon. Figure 31 (left) and [18] show the 2-across edge weave for (6^3). Notice the loops of 12 = 2 x 6 beads, and there are two beads that connect the loops of 12 beads together (across every edge). Similarly, we can imagine forming a star tiling by placing two points of the star along each edge of the interior polygon, rather than one, to produce a 2-star tiling (we thank Florence Turnour for this insight). On the right of Figure 31 (right) is the associated 2-star weave with beads on every edge and vertex. Notice the stars each have 12 points and the stars connect to neighboring stars along 2 star points.

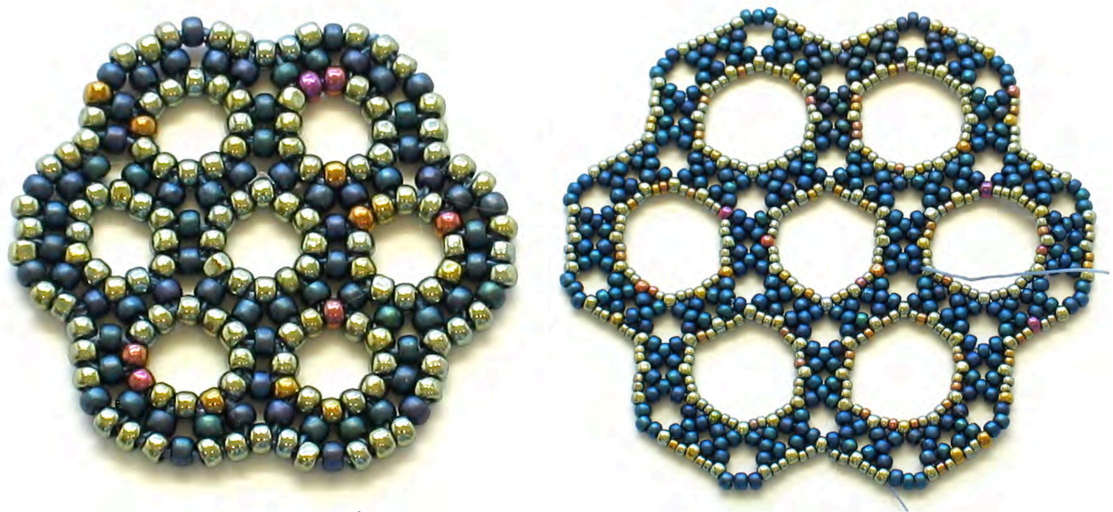
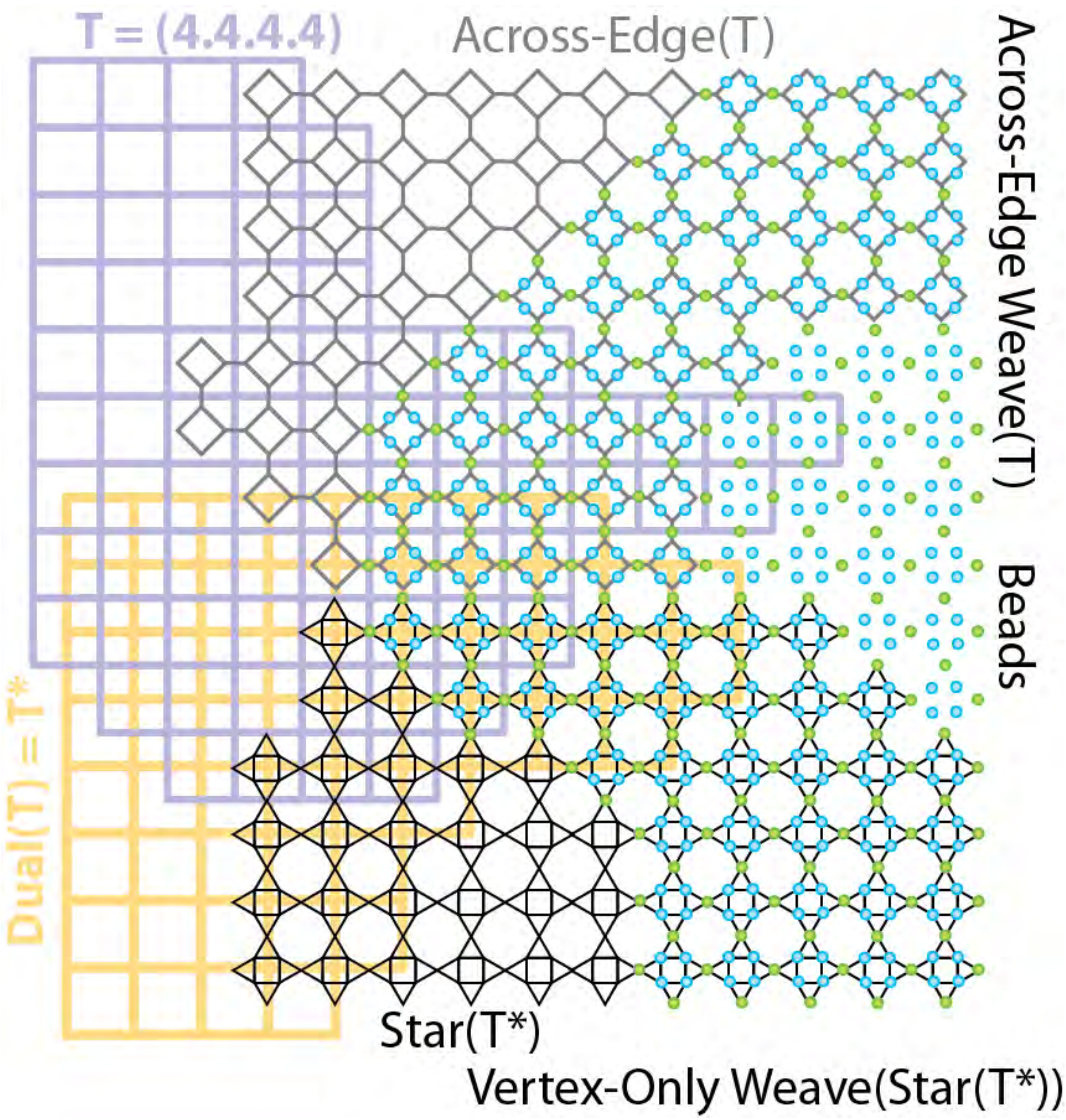


Figure 31: Theorem 3 for $T = (6^3)$ and $n = 2$, an associated 2-star weave with both vertex and edge beads (size 11° and 15° seed beads)

These constructions can be similarly defined when the across-edge beads are tripled, quadrupled, or n-tupled for any n. Theorem 3 states the corresponding generalization of Theorem 2; the proof is almost identical, so is omitted.

<u>Theorem 3:</u> Let T be a tiling, and T^* its dual tiling. Then the n-across-edge weave of T has the same pattern as the vertex-only n-star weave of T^*.

Figure 32 illustrates Theorem 3 for $T = (4^4)$ and $n = 1$ on the left and for $n = 2$ on the right.


T = (4.4.4.4)
Across-Edge(T)
Across-Edge Weave(T)
Beads
Dual(T) = T*
Star(T*)
Vertex-Only Weave(Star(T*))

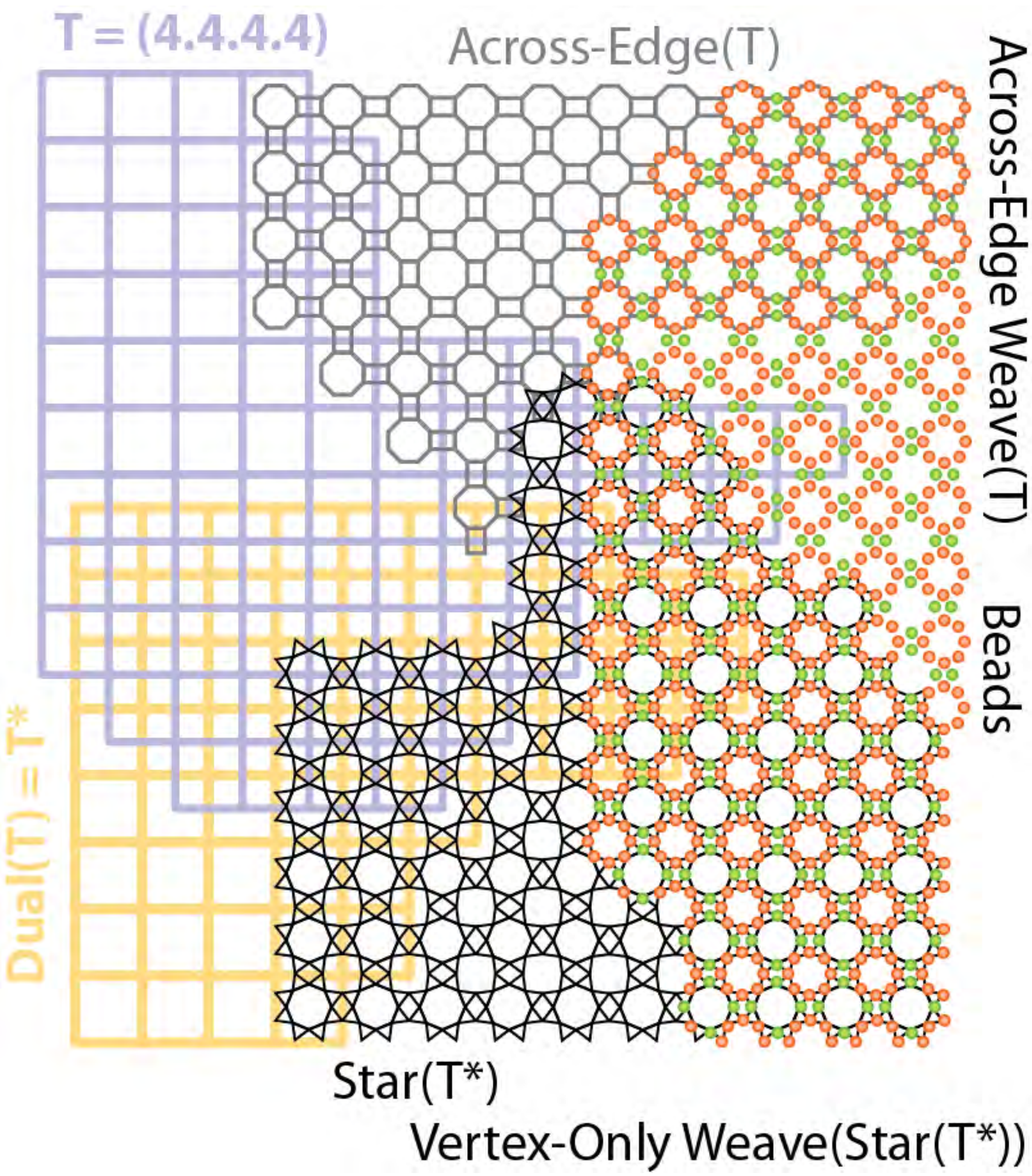


Figure 32: Theorem 3 with T = (4^4) and n =1 (left), n = 2 (right)

7. Weavable tilings

We have investigated a few of the tilings that one can realize through bead weaving. In this section we ask the more general question of which tilings one can weave with beads. We first need to explain more precisely what it means to weave a tiling with beads.

Since weaving an infinite tiling requires infinite time and money, we really mean to weave some finite piece of the tiling. We have previously used the term *patch* of a tiling informally; now we will define it as a subset of the tiles whose union is a topological disk (so bounded, connected and with no holes). A patch of a tiling is weavable if there exists an angle weave for the patch. A tiling is weavable if every patch of it is weavable. This is done by placing a bead along each edge (or at or near each vertex, or both) of the patch, and weaving a thread through the beads to fix them in their relative positions. In order to have a single piece of fabric at the end, we want to use a single, arbitrarily long, piece of thread (though it may pass through a given bead multiple times). To fix each bead in place, the thread must connect it to other beads at both ends. The continuous piece of thread through the beads corresponds to a path along the edges of the tiling. These provide the conditions for a tiling to be *weavable*.

Definition: A tiling is *(edge) weavable* when, given any patch of the tiling, there is a finite path along the edges of the tiling which passes through every edge of the patch at least once, but does not go through any edge twice in succession.

Theorem 4: Any normal tiling T has an edge-only angle weave, and therefore is edge weavable. Moreover, the thread path can be chosen to go over any edge inside the patch twice, and any edge on the boundary of the patch once.

Proof: Since any normal tiling T is locally finite, any patch of T contains only finitely many tiles. To construct the thread path, we first imagine a loop of thread around the boundary of each tile, as shown in red on the left of Figure 33.

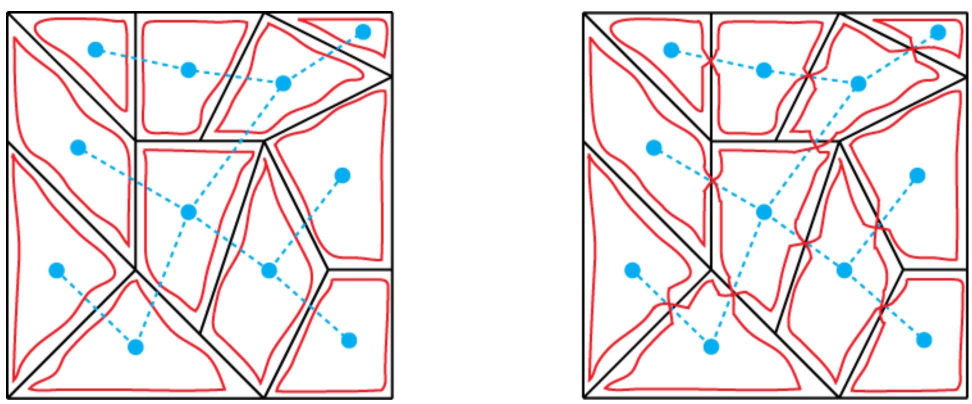


Figure 33: Constructing a thread path for a patch of a normal tiling

We also consider a *spanning tree* for the patch. More precisely, we consider a spanning tree for the dual graph that has a vertex at the center of each tile of the patch, with two tiles connected by an edge if they are adjacent. Figure 33 shows a spanning tree for the patch in blue. We now connect the loops of thread with a half-twist along each edge of the spanning tree, as on the right in Figure 33. The resulting path is a single loop of thread (since the spanning tree contains no cycles) that goes over any interior edge twice and any boundary edge once. ■

Since the class of normal tilings is infinite, Theorem 4 gives us an infinite class of weavable tilings (including, for example, all the n-star tilings of any regular tiling, for any finite n) which includes most common tilings. While our proof shows that minimal thread paths exist for weaving a given patch of a tiling with beads, in practice, most bead weavers use less optimal and more intuitive thread paths. The general method is to start with a boundary tile, and sew a loop of beads for that tile. Then advance to an adjacent tile and sew a loop of beads for that tile, which is connected to the first. The order for weaving the tiles is typically chosen so that one row of tiles is woven at a time before advancing to the next row. At the same time, some beads have very small holes and enlarging them with a file or drill is generally difficult. So it is useful to know that we can weave any patch of an edge-only angle weave with at most two passes per bead.

It is important to note that Theorem 4 only says that the tiling has an edge-only angle weave. It is easy to see that any tiling with an edge-only angle weave also has an edge-and-cover angle weave (since the thread path is the same; we're just adding more beads). However, it is not clear that any tiling that has an edge-only angle weave will also have a vertex-only or vertex-and-edge angle weave.

Fortunately, we have only discussed vertex angle weaves in the context of star tilings, and Theorem 1 implies that any star tiling T* generated from a normal tiling T *does* have a vertex-only (and hence also a vertex-and-edge) angle weave. By Theorem 4, T will have an edge-only weave, and hence an edge-and-cover weave. But by Theorem 1, T* has the same bead placement as the edge-and-cover weave of T, and the only additional thread connects the cover beads in each star. But it is clear from Figure 15 that the additional loop of thread at each star can be added without changing the orientations of any of the beads, resulting in a vertex-only angle weave of T*. This is another reason why star tilings are particularly interesting for bead weaving.

8. Areas for further consideration

We have investigated only a few of the unlimited number of beautiful beading patterns that can be created from periodic tilings. The first author and Florence Turnour are currently writing a book for bead weaving crafters that gives many examples of these patterns and explains how to weave them with beads [9]. Beyond simply weaving periodic tilings, there are many other interesting topics to be explored, such as:

- Using the colors of the beads to emphasize various patterns within the tilings.
- Creating patterns from *non*-periodic tilings, such as Penrose tilings and spiral tilings.
- Beading vertices with valence other than four.
- Determining whether every normal tiling has a vertex-only angle weave.
- Superimposing tilings to describe layered beadwork designs.
- Possible correspondences between similar star weaves (and star weaves of star weaves) generated from different tilings.
- Applying these ideas to three-dimensional objects such as polyhedra and three-dimensional space tilings.

We hope that this paper is just the beginning of a long and fruitful collaboration between the mathematics of tilings and the art of weaving beads.

Acknowledgements

We would like to thank Cindy Holsclaw and the anonymous reviewers and editors at the *Journal of Mathematics and the Arts* for their many helpful comments, including suggestions for an improved proof of Theorem 4. This paper is greatly improved because of their help.

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